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Groups with the same non-commuting graph

M.R. Darafsheh *

School of Mathematics, College of Science, University of Tehran, Tehran, Iran

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ABSTRACT

The non-commuting graph Γ_G of a non-abelian group G is defined as follows. The vertex set of Γ_G is $G - Z(G)$ where $Z(G)$ denotes the center of G and two vertices x and y are adjacent if and only if $xy \neq yx$. It has been conjectured that if G and H are two non-abelian finite groups such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$ and moreover in the case that H is a simple group this implies $G \cong H$. In this paper, our aim is to prove the first part of the conjecture for all the finite non-abelian simple groups H. Then for certain simple groups H, we show that the graph isomorphism $\Gamma_G \cong \Gamma_H$ implies $G \cong H$.

1. Introduction

Let G be a group and $Z(G)$ be its center. We will associate a graph Γ_G to G which is called the non-commuting graph of G. The vertex set $V(\Gamma_G)$ is $G - Z(G)$ and the edge set $E(\Gamma_G)$ consists of $\{x, y\}$, where x and y are distinct non-central elements of G such that $xy \neq yx$. It is clear that we are considering simple graphs, i.e. graphs with no loops or directed or repeated edges. According to [16] the non-commuting graph of a finite group G was first considered by P. Erdös in connection with the following problem. Let G be a group whose non-commuting graph Γ_G has no infinite complete subgraphs. Is it true that there is a finite bound on the cardinalities of complete subgraphs of Γ_G? In fact Neuman in [16] gave a positive answer to Erdös's question and this was the origin of many similar research about non-commuting graph of a group.

Recently in [1] some group and graph properties of the non-commuting graph associated to a non-abelian group are studied. In particular the authors put forward the following conjectures.

Conjecture 1. *Let G and H be two non-abelian finite groups such that $\Gamma_G \cong \Gamma_H$. Then $|G| = |H|$.*

Conjecture 2. *Let P be a finite non-abelian simple group and G be a group such that $\Gamma_G \cong \Gamma_P$. Then $G \cong P$.*

Since non-abelian finite simple groups are known, it is possible to prove Conjecture 1 in the case that H is a non-abelian finite simple group using a case by case consideration of H. However we prove that Conjecture 2 is related to a conjecture of Thompson which is stated below.

Conjecture 3. *If G is a finite group with $Z(G) = 1$ and M is a non-abelian finite simple group satisfying $N(G) = N(M)$, then $G \cong M$. Here the set $N(G)$ is defined by $N(G) = \{n \in \mathbb{N} \mid G$ has a conjugacy class C such that $|C| = n\}$.*

It is shown in [4] that Conjecture 3 is related to another graph associated to a finite group G, the so called Gruenberg–Kegel, or the prime graph of G. The Gruenberg–Kegel graph of a finite group G, denoted by $GK(G)$, has the set of all primes dividing the order of G as its vertex set and two distinct primes p and q are joined by an edge if and only if the

* Fax: +98 21 66412178.
E-mail address: darafsheh@ut.ac.ir.

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group G has an element of order pq. The set of all the prime divisors of G is denoted by $\pi(G)$ and the connected components of $GK(G)$ are denoted by $\pi_1, \pi_2, \ldots, \pi_{t(G)}$, where $t(G)$ denotes the number of connected components of G. In [15,17] it is proved that for any finite simple group G we have $t(G) \leq 6$. A list of all the finite simple groups G with disconnected Gruenberg–Kegel graph is available in various papers, for example one can refer to Table 1 in [5]. In [4] it is proved that Conjecture 3 is valid for simple groups G with $t(G) \geq 3$.

Our main results in this paper are:

Theorem A. *Let P be a finite non-abelian simple group. If G is a finite group with $\Gamma_G \cong \Gamma_P$. Then $|G| = |P|$.*

Theorem B. *Let P be a simple group for which the Thompson's conjecture holds. Then if G is a finite group such that $\Gamma_G \cong \Gamma_P$, we have $G \cong P$.*

Since Thompson's conjecture has been verified for all the simple groups P with $t(P) \geq 3$ in [4], so Conjecture 2 is valid for all the simple groups listed in Table II of [4]. Consequently Thompson's conjecture is valid for a variety of simple groups P with $t(P) = 2$, for example one can refer to [7] and the references quoted in that paper. In particular, since Thompson's conjecture holds for all sporadic groups, Conjecture 2 also holds for these groups as well.

Since Conjecture 2 has been verified for a special class of finite non-abelian simple groups P, Conjecture 2 is valid for this special class as well. Our notation for graphs is standard and one can consult [2] for the graph concepts that we use here. Our notation for the name finite non-abelian simple groups is obtained from [6].

2. Preliminary results

In this section we state some results which are needed to prove our main theorems. For a non-abelian group G, the non-commuting graph of G is denoted by Γ_G. The vertex set $V(\Gamma_G)$ of Γ_G is $G - Z(G)$ and two vertices x and y are joined by an edge if and only if $xy \neq yx$.

Now if H is a group and the graphs Γ_G and Γ_H are isomorphic then this means that there is a one-to-one correspondence $\varphi : G - Z(G) \longrightarrow H - Z(H)$ preserving edges, i.e. if $x, y \in G - Z(G)$ and $xy \neq yx$ then $\varphi(x)\varphi(y) \neq \varphi(y)\varphi(x)$. Equivalently if we consider the complimentary graphs of Γ_G and Γ_H, then we have the condition that $x, y \in G - Z(G)$ and $xy = yx$ implies $\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$. The graph isomorphism $\Gamma_G \cong \Gamma_H$ implies also $|G - Z(G)| = |H - Z(H)|$, and if we assume only G is non-abelian, then $|G - Z(G)| \neq 0$ implying $|H - Z(H)| \neq 0$ and consequently H must be non-abelian. Since $|Z(H)| \leq |H - Z(H)| = |G - Z(G)|$, finiteness of G implies that $Z(H)$ and consequently H is a finite group. Therefore if in the isomorphism $\Gamma_G \cong \Gamma_H$ we assume G is a non-abelian finite group, then the same conditions will be satisfied by H.

The degree of a vertex v in a graph Γ is defined to be the number of edges adjacent to v and it is denoted by $d(v)$. It is easy to see that the degree of a vertex g in the non-commuting graph Γ_G of G is equal to $d(G) = |G| - |C_G(g)|$ which can be written as $d(G) = |C_G(g)|\,([G : C_G(g)] - 1)$. Therefore both $|C_G(g)|$ and $Z(G)$ are divisors of $d(G)$. In the following P is a finite non-abelian group and G is a group such that $\Gamma_P \cong \Gamma_G$ is a graph isomorphism.

Lemma 1. *Let $\Gamma_P \cong \Gamma_G$. Then for any non-empty subset S of P which intersects $Z(P)$ in empty set, $|Z(G)|$ divides $|C_P(S)| - |Z(P)|$.*

Proof. By definition from $\Gamma_P \cong \Gamma_G$ it follows that there is a one-to-one correspondence $\varphi : P - Z(P) \longrightarrow G - Z(G)$ preserving edges of the graphs. Now any subset S of P with $Z(P) \cap S = \phi$ the bijection φ induces the one-to-one correspondence $\varphi : C_P(S) - Z(P) \longrightarrow C_G(\varphi(S)) - Z(G)$. Therefore $|C_P(S)| - |Z(P)| = |C_G(\varphi(S))| - |Z(G)|$, from which it follows that $|C_P(S)| - |Z(P)| = |Z(G)|\,([C_G(\varphi(S)) : Z(G)] - 1)$, consequently $|Z(G)| \mid |C_P(S)| - |Z(P)|$ and the Lemma follows. ■

We remark that if $S = \{x\}, x \notin Z(P)$, then $|Z(G)| \mid |C_P(x)| - |Z(P)|$ which is mentioned in Lemma 3.1 of [1]. Obviously from $\Gamma_P \cong \Gamma_G$ it is evident that $|P| - |Z(P)| = |G| - |Z(G)|$ from which it follows that $|Z(G)| \mid |P| - |Z(P)|$ and consequently $Z(G)$ divides $|C_P(x)| - |Z(P)|$, hence $|Z(G)| \mid |P| - |C_P(x)|$ for all $x \in P - Z(P)$. Hence $|Z(G)| \mid |C_P(x)|\,([P : C_P(x)] - 1)$. If P is a centerless group, i.e. $|Z(P)| = 1$, then $|Z(G)| \mid |C_P(x)| - 1$ implies that $|Z(G)|$ and $|C_P(x)|$ are coprime and from $|Z(G)| \mid |C_P(x)|\,([P : C_P(x)] - 1)$ we obtain $|Z(G)| \mid \left|x^P\right| - 1$, where x^P is the conjugacy class in P containing x.

Lemma 2. *Let P and G have trivial centers and $\varphi : \Gamma_P \longrightarrow \Gamma_G$ be a graph isomorphism. Then for any $x \in P$ we have $\varphi(C_P(x)^*) = C_G(\varphi(x))^*$, where A^* is $A - \{1\}$.*

Proof. By definition $\varphi : P - \{1\} \longrightarrow G - \{1\}$ is a one-to-one correspondence preserving edges. For any $y \in \varphi(C_G(x)^*)$ we have $y = \varphi(t), t \in C_G(x)^*$, hence $tx = xt$ implying $\varphi(t)\varphi(x) = \varphi(x)\varphi(t)$ or $y\varphi(x) = \varphi(x)y$. Therefore $y \in C_G(\varphi(x))^*$ proving $\varphi(C_P(x)^*) \subseteq C_G(\varphi(x))^*$.

Conversely if $y \in C_G(\varphi(x))^*$, then $y \in G$ and $y\varphi(x) = \varphi(x)y$. Since φ is onto there is $g \in P$ such that $\varphi(g) = y$. Hence $\varphi(g)\varphi(x) = \varphi(x)\varphi(g)$ implying $gx = xg$ or $g \in C_P(x)^*$, giving $y \in \varphi(C_P(x)^*)$. Therefore $C_G(\varphi(x))^* \subseteq \varphi(C_P(x)^*)$ and the Lemma is proved. ■

Before stating the next lemma we define the following concept. For a finite group G we define $N(G) = \{n \in \mathbb{N} \mid$ there is a conjugacy class of G with size $n\}$.

Lemma 3. *If $\Gamma_P \cong \Gamma_G$ and P and G are centerless, then $N(P) = N(G)$.*

Proof. As in the proof of Lemma 2 let $\varphi : P - \{1\} \longrightarrow G - \{1\}$ be a one-to-one correspondence of the vertices of the graphs Γ_P and Γ_G with the extension $\varphi(1) = 1$. In this manner we also have $|P| = |G|$. For $x \in P$ let x^P denote the conjugacy class of x in P. Using Lemma 2 we do the following calculation: $\left|x^P\right| = [P : C_P(x)] = \frac{|P|}{|C_P(x)|} = \frac{|G|}{|\varphi(C_P(x))|} = \frac{|G|}{|C_G(\varphi(x))|} = [G : C_G(\varphi(x))] = \left|(\varphi(x))^G\right|$. Since φ is a bijection we observe that there is a one-to-one correspondence between the conjugacy class sizes of P and G, hence $N(P) = N(G)$. ■

3. Main results

Given a finite non-abelian simple group P, we are concerned with a group G satisfying $\Gamma_P \cong \Gamma_G$. As a matter of fact we want to prove $|P| = |G|$ and if possible $P \cong G$. In [1] the following results related to above questions are obtained. For the alternating group $\mathbb{A}_n$, $n \geq 5$, the projective special linear groups $PSL_{2^k}(2)$ and $PSL_2(q)$, q odd, it is proved that if P is any of these groups and if G is a group satisfying $\Gamma_P \cong \Gamma_G$, then $|P| = |G|$. Furthermore it is also proved that if P is any of the simple groups $PSL_2(2^n)$ or the Suzuki simple groups $Suz(2^{2n+1})$, $n > 1$, then $\Gamma_P \cong \Gamma_G$ implies $P \cong G$. As far as the author is aware no results of this kind have been published so far. Therefore first of all we show that if P is a non-abelian finite simple group and if the group G satisfies $\Gamma_P \cong \Gamma_G$, then $|P| = |G|$. Our notation for the names of the finite simple groups is the same as used in [6]. In what follows for the proof of Theorem 1 we need to prove some common property of the simple groups of Lie type. For the terminology and basic properties of these groups one can refer to [3].

Lemma 4. *Let P be a simple group of Lie type defined over a field of q elements where q is a power of the prime p. If S is a Sylow p-subgroup of P, then the centralizer of S in P is non-trivial and is contained in S.*

Proof. Let B the Borel subgroup containing S, so that $B = N_P(S)$ and $B = SH$, where H is a Cartan subgroup; $C_P(S) \subset N_P(S) = B$. Also H is abelian of order prime to p, so that $C_P(S) = Z(S).Y$, where Y is a subgroup of H. By ([3], p. 20) there is w in the Weyl group W which transforms every positive root to a negative root. Therefore the corresponding element n_w of P which is denoted again by w transforms S to its opposite, i.e. $S^w = w^{-1}Sw$ is generated by the root subgroups X_{-r}, where X_r is a generator of S. By ([3], p. 68) we have $P = \langle S, S^w \rangle$. By ([3], p. 102) $w^{-1}h(\chi)w = h(\chi')$ where $\chi'(s) = \chi(w^{-1}(s))$; here $\chi(w^{-1}(s)) = \chi(-s) = \chi(s)^{-1}$ so for all $h \in H$ and a fortiori for all $y \in Y$ we have $w^{-1}hw = h^{-1}$. From $C_P(S) = Z(S).Y$ we see that each element of Y centralizes S and from $w^{-1}yw = y^{-1}$ we obtain for all $s \in S$:

$$y^{-1}s^w = y^{-1}w^{-1}sw = w^{-1}ysw = w^{-1}syw = w^{-1}swy^{-1} = s^w y^{-1}$$

which implies that y^{-1} centralizes S^w; hence each element of Y centralizes S^w. Since Y centralizes both S and S^w this forces Y to be in the center of P. But, P being a simple group, its center is trivial, and this forces Y to be trivial as well. This shows that $C_P(S) = Z(S)$. ■

Theorem 1. *Let P be a finite non-abelian simple group. If G is a group such that $\Gamma_P \cong \Gamma_G$, then $|P| = |G|$.*

Proof. According to the classification of finite simple groups we have one of the following possibilities for P: an alternating group $\mathbb{A}_n$, $n \geq 5$; a group of Lie type; or one of the 26 sporadic groups. Therefore we deal with the above possibilities of P separately. The case of $P \cong \mathbb{A}_n$, $n \geq 5$ has been dealt with in [1]. Therefore we deal with the other cases. But first note that $Z(P) = 1$ and from $\Gamma_P \cong \Gamma_G$ it follows that $|P| - 1 = |G| - |Z(G)|$, hence if we prove $Z(G) = 1$, then it follows that $|P| = |G|$. We continue with the following cases:

If P is isomorphic to one of the 26 sporadic simple groups, by the remark made after Lemma 1, $|Z(G)|$ is a divisor of $|P| - 1$ and $|C_P(x)| - 1$ for any $x \in P$. One can observe (cf [6]) the following common property of sporadic groups: If P is such a group, let p be the largest prime divisor of $|P|$; then (i) $p - 1$ divides $|P|$ (ii) there exist $x \in P$ of order p such that $C_P(\langle x \rangle) = \langle x \rangle$. Now since $|Z(G)| \mid |P| - 1$ we deduce $(|Z(G)|, |P|) = 1$. From $|Z(G)| \mid p - 1$ and $p - 1 \mid |P|$ we obtain $|Z(G)| \mid |P|$, hence $|Z(G)| = (|Z(G)|, |P|) = 1$ proving $|Z(G)| = 1$.

In the following, we will consider all the simple groups of Lie type. Let P be a simple group of Lie type defined over a field with q elements where q is a power of the prime p. By Lemma 4 if S is a Sylow p-subgroup of P, then $C_P(S)$ is non-trivial and is contained in S. In particular if $|C_P(S)| = q^t$, then $t > 0$ and $q^t - 1 \mid |P|$. By Lemma 1 we have $|Z(G)| \mid |C_P(S)| - 1 = q^t - 1$, hence $|Z(G)| \mid |P|$. But $|Z(G)|$ and $|P|$ are relatively prime from which it follows that $|Z(G)| = 1$ and the Theorem is proved. ■

Theorem 2. *Let P be a finite non-abelian simple group for which the Thompson's conjecture holds. Then if G is a finite group such that $\Gamma_P \cong \Gamma_G$, we have $G \cong P$.*

Proof. From $\Gamma_P \cong \Gamma_G$ and using Theorem 1 we obtain $Z(G) = 1$. Now since G and P are centerless, by Lemma 3 it follows that $N(G) = N(P)$. Since Thompson's conjecture holds for P we deduce $G \cong P$ and the Theorem is proved. ■

Corollary 1. *Conjecture 2 holds for any of the following simple groups.*

(a) *A sporadic simple group,*

(b) *Any simple group with at least 3 prime graph components,*

(c) *A few of simple groups with 2 prime graph components such as:* $A_{p-1}(q)$; ${}^2A_{p-1}(q)$; ${}^2D_p(3)$, $p \geq 5$ *a prime number not of the form* $2^m + 1$; $L_{p+1}(2)$; $E_6(q)$; ${}^2E_6(q)$; ${}^2D_n(q)$, $n = 2^m \geq 4$.

Proof. By [4,5,7,8,10–14] Thompson's conjecture holds for the above groups, hence the result follows by Theorem 2. ■

In [1], Theorem 3.16, p. 482, it is also proved that for $n > 2$, the graph isomorphism $\Gamma_{\mathbb{S}_n} \cong \Gamma_G$ implies $|\mathbb{S}_n| = |G|$, where $\mathbb{S}_n$ denotes the symmetric group of degree n. We will show that if P is a non-abelian simple group and $A = \mathrm{Aut}(P)$ then $\Gamma_A \cong \Gamma_G$ will imply $|A| = |G|$ for certain classes of simple groups P. Of course if $n \neq 2, 6$, then $\mathrm{Aut}(\mathbb{A}_n) \cong \mathbb{S}_n$, hence we obtain a generalization of the above result of [1].

Theorem 3. *Let P be a simple alternating or a simple sporadic group or one of the following simple groups of Lie type: $A_n(q), B_n(q), C_n(q), {}^2D_n(q), n \geq 4, F_4(q), G_2(q)$. Let $A = \mathrm{Aut}(P)$. If G is a group such that $\Gamma_A \cong \Gamma_G$, then $|A| = |G|$.*

Proof. Since $Z(A)$ is trivial it is enough to show that $Z(G)$ is also trivial. If $P \cong \mathbb{A}_n$, $n \neq 6$, $n \geq 5$, the result follows by [1]. For $P = \mathbb{A}_6 \cong PSL_2(9)$ we have $A = \mathrm{Aut}(P) \cong P\Gamma L_2(9)$. Where $P\Gamma L_2(9) = \frac{\Gamma L_2(9)}{Z(\Gamma L_2(9))}$ and $\Gamma L_2(9)$ is the group of all the semi-linear transformation of the vector space of dimension 2 over the Galois field with q element. Using [6] we observe that $A = P\Gamma L_2(9)$ has elements x and y of orders 2 and 5, respectively such that $|C_A(x)| = 24$ and $|C_A(y)| = 5$. By Lemma 1 we must have $|Z(G)| \mid |C_A(x)| - 1 = 23$ and $|Z(G)| \mid |C_A(y)| - 1 = 4$, implying $Z(G) = 1$.

If P is a sporadic group, then there exists $x \in P$ such that $C_A(x) = \langle x \rangle$ (Theorem 3.4 p. 113 of [9]); more precisely:

(a) If $P \neq J_2, M^cL$, let p be the greatest prime divisor of $|P|$, then there exists $x \in P$ of order p such that $C_P(x) = \langle x \rangle$ and $p - 1$ divides $|P|$. Moreover, by the proof of Theorem 3.4 of [9], $C_P(x) = C_A(x)$ and $p - 1$ divides $|P|$, therefore $p - 1$ divides $|A|$. Now Lemma 1 implies $Z(G) = 1$.

(b) If $P = J_2$, the proof of Theorem 3.4 of [9] gives $x \in P$ of order 15 such that $C_P(x) = C_A(x) = \langle x \rangle$; since $15 - 1 = 14$ divides $|J_2|$ we are done.

(c) If $P = M^cL$, the proof of Theorem 3.4 of [9] gives $x \in P$ of order 14 such that $C_P(x) = C_A(x) = \langle x \rangle$. But by our observation after the proof of Lemma 1, we must have $|Z(G)| \mid |C_A(x)| - 1 = 13$ and $|Z(G)| \mid |A| - 1$. But $|A| = 2\,|M^cL| = 2^8.3^6.5^3.7.11$ and $|A| - 1$ is prime to 13, hence $|Z(G)| = 1$ and we are done.

Finally, assume P is a simple group of Lie type. By Theorem 3.1 page 107 of [9], if $P \neq {}^2A_3(2), D_4(q), q \leq 5$, there exists a torus T of A such that $C_A(T \cap P) = T$. But T is abelian and $C_A(T) \subset C_A(T \cap P) = T$, hence $C_A(T) = T$: moreover $|T|$ is given in Table III, page 108–109 of [9]. More precisely:

If $P = A_n(q)$, then $|T| = \frac{q^{n+1}-1}{q-1}$,

If $P = B_n(q), C_n(q), {}^2D_n(q), n \geq 4$, then $|T| = q^n + 1$,

If $P = F_4(q)$, then $|T| = q^4 + 1$,

If $P = G_2(q)$, then $|T| = q^2 + q + 1$.

In all of the above cases clearly $|T| - 1$ divides $|P|$. Hence in every case $|T| - 1$ divides $|A|$. Therefore $|Z(G)| \mid |A|$ as well as $|Z(G)| \mid |A| - 1$. Hence $|Z(G)| = 1$ and the Theorem is proved.

Corollary 2. *Let P be a sporadic group, except M^cL and J_2, or one of the following simple groups of Lie type: $A_{p-1}(q)$; ${}^2D_p(3)$, $p \geq 5$ a prime number not of the form $2^m + 1$; ${}^2A_{p-1}(q)$, p an odd prime number; $C_n(q)$, $n = 2^m \geq 2$, q a power of 2; ${}^2D_n(q)$, $n = 2^m \geq 4$; $F_4(q)$, q odd; $G_2(q)$, $q \equiv \epsilon \pmod 3$, $\epsilon = \pm 1$, $q > 2$. Let $A = \mathrm{Aut}(P)$. If G is a centerless group such that $\Gamma_A \cong \Gamma_G$, then $A \cong G$.*

Proof. Clearly A is a centerless group. Since G is assumed to be centerless we have $Z(G) = 1$, and from $\Gamma_A \cong \Gamma_G$ by Lemma 3 we obtain $N(A) = N(G)$. Since Thompson's conjecture is true for all automorphism groups of the simple groups mentioned in the corollary, by Theorem 2 the result follows. ■

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